Tensor products of *q*-superalgebra representations and *q*-series identities

1997 J. Phys. A: Math. Gen. 30 7147

(http://iopscience.iop.org/0305-4470/30/20/017)

J. Phys. A: Math. Gen. **30** (1997) 7147–7166. Printed in the UK PII: S0305-4470(97)84399-5

Tensor products of q-superalgebra representations and q-series identities

Won Sang Chung†, E G Kalnins‡ and W Miller Jr§

† Department of Physics, Gyeongsang National University, Jinju, 660-701, Korea
‡ Department of Mathematics and Statistics, University of Waikato, Hamilton, New Zealand
§ School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

Received 20 May 1997

Abstract. We work out examples of tensor products for distinct q-generalizations of Euclidean, oscillator and $s\ell(2)$ type superalgebras in cases where the method of highest-weight vectors will not apply. In particular, we use the three-term recurrence relations for Askey–Wilson polynomials to decompose the tensor product of representations from the positive discrete series and representations from the negative discrete series. We show that various q-analogues of the exponential function can be used to mimic the exponential mapping from a Lie algebra to its Lie group and we compute the corresponding matrix elements of the 'group operators' on these representation spaces. We show that the matrix elements themselves transform irreducibly under the action of the quantum superalgebra. The most important q-series identities derived here are interpreted as the expansion of the matrix elements of a 'group operator' (via the exponential mapping) in a tensor product basis in terms of the matrix elements in a reduced basis. They involve q-hypergeometric series with base $-q$, $0 < q < 1$.

1. Introduction

Zhedanov and others have introduced a product of generalized $s\ell_q(2)$ algebras that allows one to take tensor products of representations corresponding to two distinct algebras [7, 8, 28]. Their generalization is an algebra (v, u) with generators H, E_+, E_- which obey the commutation relations

$$[H, E_+] = E_+ \qquad [H, E_-] = -E_- \qquad [E_+, E_-] = -uq^{-H} - vq^{H}. \tag{1}$$

Here, u and v are real numbers and $0 < q < 1$. For $uv \neq 0$ this algebra is isomorphic to one of the true $s\ell_q(2)$ type algebras, for $uv = 0$, $u^2 + v^2 > 0$ it is isomorphic to a special realization of the q-oscillator algebra, and for $u = v = 0$ it is isomorphic to the Euclidean Lie algebra $m(2)$ [12]. This algebra has an invariant element

$$C = E_+E_- + \frac{vq^H - uq^{1-H}}{1-q}. \tag{2}$$

As pointed out by Zhedanov and others [7, 8, 28], the family of algebras admits a multiplication $(v, u) \otimes (-u, t) \cong (v, t)$, defined by

$$\begin{aligned} F_+ &= \Delta(E_+) = E_+ \otimes q^{\frac{1}{2}H} + q^{-\frac{1}{2}H} \otimes E_+ \\ F_- &= \Delta(E_-) = E_- \otimes q^{\frac{1}{2}H} + q^{-\frac{1}{2}H} \otimes E_- \\ L &= \Delta(H) = H \otimes I + I \otimes H. \end{aligned} \tag{3}$$

The operators $F_\pm$, L satisfy the commutation relations (1). Using (3) we can easily define the tensor product $\rho \otimes \mu$ of a representation ρ of (v, u) and the representation μ of $(-u, t)$, thereby obtaining a representation of (v, t). This construction yields a convenient generalization of the tensor product computations in, for example, [13–15]. We follow this idea to study generalizations of the $osp_q(1/2)$ algebra [3, 21, 23].

In this paper the q-superalgebra $[v, u]$ is defined by the generators $H, V_\pm$ and the relations

$$[H, V_\pm] = \pm\tfrac{1}{2}V_\pm \qquad \{V_+, V_-\} = -uq^{-2H} - vq^{2H} \tag{4}$$

where $[A, B] = AB - BA$ and $\{A, B\} = AB + BA$. Here u, v are parameters. The 'Casimir' operator C for $[v, u]$ is

$$C = V_+V_- + \frac{v}{1+q}q^{2H} + \frac{u}{1+q^{-1}}q^{-2H} \tag{5}$$

and satisfies the relations

$$[H, C] = 0 \qquad \{V_\pm, C\} = 0. \tag{6}$$

Note that C^2 is an invariant operator, i.e. C^2 commutes with all elements of the $[v, u]$ algebra.

We define the coproduct

$$\begin{aligned} L &= \Delta(H) = H \otimes I + I \otimes H \\ F_\pm &= \Delta(V_\pm) = V_\pm \otimes q^H + q^{-H} \otimes V_\pm \end{aligned} \tag{7}$$

where

$$(A \otimes B)(C \otimes D) = (-1)^{p(B)p(C)} AC \otimes BD. \tag{8}$$

Here $p(A)$ is the *parity* of the operator A. In this case the bosonic variables H, q^H have parity 0 and the fermionic variables $V_\pm$ have parity 1 [2, 21]. It follows that this algebra admits a multiplication of the form

$$[v, u] \otimes [-u, t] = [v, t] \tag{9}$$

where the first factor corresponds to the algebra $[v, u]$ and the second factor to algebra $[-u, t]$. Indeed the relations

$$[L, F_\pm] = \pm\tfrac{1}{2}F_\pm \qquad \{F_+, F_-\} = -tq^{-2L} - vq^{2L} \tag{10}$$

are satisfied. The only non-trivial part of the proof is

$$\begin{aligned} \{F_-, F_+\} &= (-uq^{-2H} - vq^{2H}) \otimes q^{2H} + q^{-2H} \otimes (-tq^{-2H} + uq^{2H}) \\ &= -tq^{-2H} \otimes q^{-2H} - vq^{2H} \otimes q^{2H} = -tq^{-2L} - vq^{2L}. \end{aligned}$$

In sections 2 and 3 we study irreducible representations of the 'Euclidean' algebra [0, 0] and work out the (non-unique) tensor product decomposition for

$$[0, 0] \otimes [0, 0] \cong [0, 0].$$

We compute the Clebsch–Gordan coefficients for the expansion and use them to derive q-series identities for the special functions that appear naturally in the theory. They are interpreted here as expansions of the matrix elements of a 'group operator' in a tensor product basis in terms of the matrix elements in a reduced basis. The q-series are base $-q$, as follows naturally from expressions (7). In section 4 we carry out the analogous constructions for positive discrete series representations of the general algebra $[v, u]$ for $v^2 + u^2 > 0$ tensored with negative discrete series representations. Here our methods lead naturally to a three-term recurrence relation for the Clebsch–Gordan coefficients that can be solved through comparison with the recurrence relation with Askey–Wilson polynomials. This yields the measure (not necessarily positive) determining the decomposition of the tensor product into irreducible components.

The notation used for q-series in this paper follows that of Gasper and Rahman [6].

2. Euclidean q-superalgebra representations

The three dimensional q-supersymmetric Lie algebra $[0, 0]$ is determined by its generators H, V_+, V_- which obey the relations

$$[H, V_\pm] = \pm V_\pm \qquad \{V_+, V_-\} = 0. \tag{11}$$

We consider an analogy (ω) of the infinite dimensional irreducible representations of the Euclidean Lie algebra, characterized by the non-zero complex number ω The spectrum of H corresponding to (ω) is the set $S = \{n/2 : n \in Z\}$ and the complex representation space has basis vectors f_m, $m \in S$, such that

$$V_+ f_m = \omega f_{m+1} \qquad V_- f_m = (-1)^m \omega f_{m-1} \qquad H f_m = \frac{m}{2} f_m \tag{12}$$

where $C \equiv V_+V_-$ is the 'Casimir' operator, with $C f_m = (-1)^m \omega^2 f_m$. Note that the representations (ω) and $(-\omega)$ are equivalent.

A simple realization of (ω) is given by the operators

$$H = \frac{1}{2} z \frac{\mathrm{d}}{\mathrm{d}z} \qquad V_+ = \omega z \qquad V_- = \frac{\omega}{z} R_z \tag{13}$$

acting on the space of all linear combinations of the functions z^n, z a complex variable, $n \in Z$, with basis vectors $f_m(z) = z^m$. Here, the operator R_z acts on functions $f(z)$ to give $R_z f(z) = f(-z)$. Note that

$$q^H = T_z^{1/2} \qquad \text{where } \; T_z^\alpha f(z) = f(q^\alpha z).$$

We can introduce an inner product on the dense subspace of all finite linear combinations of the basis vectors, such that $\langle f_n, f_{n'}\rangle = \delta_{nn'}$, $n, n' \in Z$. This induces the representation $(\overline{\omega})^*$ of $[0, 0]$, adjoint to (ω), defined by the operators $V'_\pm = V^*_\mp$, $H' = H = H^*$ such that

$$V'_+ f_m = (-1)^{m+1} \overline{\omega} f_{m+1} \qquad V'_- f_m = \overline{\omega} f_{m-1} \qquad H' f_m = \frac{m}{2} f_m. \tag{14}$$

For ω real, this means $V'_+ = (-1)^{2H} V_+$, $V'_- = V_-(-1)^{2H}$.

In terms of the operators (13) we can obtain a realization of (ω) and its Hilbert space structure by setting $z = \mathrm{e}^{i\theta}$:

$$\begin{aligned} &H = -\frac{i}{2} \frac{\mathrm{d}}{\mathrm{d}\theta} \qquad V_+ = \omega \mathrm{e}^{i\theta} \qquad V_- = \omega \mathrm{e}^{-i\theta} R_z \\ &f_n(z) = \mathrm{e}^{in\theta} \qquad \langle f, f'\rangle = \frac{1}{2\pi} \int_0^{2\pi} f(\mathrm{e}^{i\theta}) \overline{f'(\mathrm{e}^{i\theta})}\, \mathrm{d}\theta. \end{aligned} \tag{15}$$

With the q-analogues of the exponential function

$$\begin{aligned} &e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k} = \frac{1}{(x;q)_\infty} \qquad |x| < 1 \\ &E_q(x) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} x^k}{(q;q)_k} = (-x;q)_\infty \end{aligned} \tag{16}$$

we employ the model (13) to define the following q-analogues of matrix elements of (ω):

$$\text{(a)}\qquad e_{-q}(\beta V_+)E_{-q}(\alpha V_-)f_n = \sum_{n'=-\infty}^{\infty} T_{n'n}^{(e+,E-)}(\alpha,\beta)f_{n'} \qquad |\omega\beta| < 1 \tag{17}$$

$$\text{(b)}\qquad E_{-q}(\beta V_+)e_{-q}(\alpha V_-)f_n = \sum_{n'=-\infty}^{\infty} T_{n'n}^{(E+,e-)}(\alpha,\beta)f_{n'} \qquad |\omega\alpha| < 1. \tag{18}$$

$$\text{(c)}\qquad e_{-q}(\beta V_-)E_{-q}(\alpha V_+)f_n = \sum_{n'=-\infty}^{\infty} T_{n'n}^{(e-,E+)}(\alpha,\beta)f_{n'} \qquad |\omega\beta| < 1 \tag{19}$$

$$\text{(d)}\qquad E_{-q}(\beta V_-)e_{-q}(\alpha V_+)f_n = \sum_{n'=-\infty}^{\infty} T_{n'n}^{(E-,e+)}(\alpha,\beta)f_{n'} \qquad |\omega\alpha| < 1 \tag{20}$$

where $0 < q < 1$ and $\alpha, \beta \in \mathbb{C}$. (Here, α, β are *not* fermionic variables which is the usual choice in defining the action of supersymmetric groups [20, 25].) The motivation for the choice of base $-q$ comes from the relations

$$(V_\pm \otimes q^H)(q^{-H} \otimes V_\pm) = -q^{\pm 1}(q^{-H} \otimes V_\pm)(V_\pm \otimes q^H) \tag{21}$$

and the well known property that if X and Y are linear operators such that $YX = pXY$ then [6, page 28]

$$(Y+X)^k = \sum_{\ell=0}^{k} \frac{(p;p)_k}{(p;p)_\ell (p;p)_{k-\ell}} X^\ell Y^{k-\ell}$$
$$e_p(X+Y) = e_p(X)e_p(Y) \qquad E_p(X+Y) = E_p(Y)E_p(X). \tag{22}$$

Since $V_+^* = V_-'$, $V_-^* = V_+'$ we have

$$T_{n'n}^{(e\pm,E\mp)}(\alpha,\beta) = \overline{T'^{(E\pm,e\mp)}_{nn'}(\overline{\beta},\overline{\alpha})}. \tag{23}$$

$$T_{n'n}^{(E\pm,e\mp)}(\alpha,\beta) = \overline{T'^{(e\pm,E\mp)}_{nn'}(\overline{\beta},\overline{\alpha})}. \tag{24}$$

Furthermore, since $e_{-q}(x)E_{-q}(-x) = 1$, we have the identities

$$\sum_{\ell=-\infty}^{\infty} T_{n\ell}^{(e-,E+)}(\alpha,\beta)T_{\ell n'}^{(e+,E-)}(-\beta,-\alpha) = \delta_{nn'} \qquad |\omega\alpha|, |\omega\beta| < 1 \tag{25}$$

$$\sum_{\ell=-\infty}^{\infty} T_{n\ell}^{(E+,e-)}(\alpha,\beta)T_{\ell n'}^{(E-,e+)}(-\beta,-\alpha) = \delta_{nn'} \qquad |\omega\alpha|, |\omega\beta| < 1. \tag{26}$$

(Note that our operator derivations of these formulae and of some formulae to follow lead automatically to formal power series identities in the 'group parameters'. These identities must then be examined case by case to determine when the series are convergent as analytic functions of the group parameters.) Using the model (13) to treat (17)–(20) as generating functions for the matrix elements and computing the coefficients of $z^{n'}$ in the resulting expressions we obtain the explicit results ($p = -q$):

$$T_{n'n}^{(e+,E-)}(\alpha,\beta) = \frac{(\beta\omega)^{n'-n}}{(p;p)_{n'-n}}\left[{}_1\phi_1\left(\begin{matrix}0\\ p^{n'-n+1}\end{matrix};p,(-1)^{n+1}i\alpha\beta\omega^2\right)\frac{1-i}{2}\right.$$
$$\left. + {}_1\phi_1\left(\begin{matrix}0\\ p^{n'-n+1}\end{matrix};p,(-1)^{n}i\alpha\beta\omega^2\right)\frac{1+i}{2}\right] \tag{27}$$

$$T_{n'n}^{(E+,e-)}(\alpha,\beta) = \frac{(\beta\omega)^{n'-n} p^{(n'-n)(n'-n-1)/2}}{(p;p)_{n'-n}}$$
$$\times \left[{}_1\phi_1\left(\begin{matrix} 0 \\ p^{n'-n+1} \end{matrix} ; p, (-1)^{n+1} p^{n'-n} i\alpha\beta\omega^2 \right) \frac{1-i}{2} \right.$$
$$\left. + {}_1\phi_1\left(\begin{matrix} 0 \\ p^{n'-n+1} \end{matrix} ; p, (-1)^{n} p^{n'-n} i\alpha\beta\omega^2 \right) \frac{1+i}{2} \right] \tag{28}$$

$$T_{n'n}^{(e-,E+)}(\alpha,\beta) = \frac{(\alpha\omega)^{n'-n} p^{(n'-n)(n'-n-1)/2}}{(p;p)_{n'-n}} \left[{}_1\phi_1\left(\begin{matrix} 0 \\ p^{n'-n+1} \end{matrix} ; p, (-1)^{n'} p^{n'-n} i\alpha\beta\omega^2 \right) \frac{1-i}{2} \right.$$
$$\left. + {}_1\phi_1\left(\begin{matrix} 0 \\ p^{n'-n+1} \end{matrix} ; p, (-1)^{n'+1} p^{n'-n} i\alpha\beta\omega^2 \right) \frac{1+i}{2} \right] \tag{29}$$

$$T_{n'n}^{(E-,e+)}(\alpha,\beta) = \frac{(\alpha\omega)^{n'-n}}{(p;p)_{n'-n}} \left[{}_1\phi_1\left(\begin{matrix} 0 \\ p^{n'-n+1} \end{matrix} ; p, (-1)^{n'} i\alpha\beta\omega^2 \right) \frac{1-i}{2} \right.$$
$$\left. + {}_1\phi_1\left(\begin{matrix} 0 \\ p^{n'-n+1} \end{matrix} ; p, (-1)^{n'+1} i\alpha\beta\omega^2 \right) \frac{1+i}{2} \right]. \tag{30}$$

Here we have made use of the identity

$$(-1)^{n(n-1)/2} = \frac{1-i}{2}(i)^n + \frac{1+i}{2}(-i)^n. \tag{31}$$

These results make sense for $n > n'$ as well as $n' \geqslant n$. Indeed, for $k > 0$ we have

$$\lim_{m\to -k} \frac{1}{(p;p)_m} {}_1\phi_1\left(\begin{matrix} 0 \\ p^{m+1} \end{matrix} ; p, B \right) = \frac{p^{k(k-1)/2}(-B)^k}{(p;p)_k} {}_1\phi_1\left(\begin{matrix} 0 \\ p^{k+1} \end{matrix} ; p, p^k B \right). \tag{32}$$

The matrix elements $T_{n'n}(\alpha,\beta)$ themselves define models of the representations (ω). We can see this directly from (11). For example, it is a simple consequence of these relations and $e_p(x) = (x;p)_\infty^{-1}$, $E_p(x) = (-x;p)_\infty$ that

$$e_p(\beta V_+)E_p(\alpha V_-)V_+ = \frac{1}{\beta}(I - T_\beta)R_\alpha e_p(\beta V_+)E_p(\alpha V_-) \tag{33}$$
$$e_p(\beta V_+)E_p(\alpha V_-)V_- = \frac{p}{\alpha}(T_\alpha^{-1} - I)e_p(\beta V_+)E_p(\alpha V_-) \tag{34}$$

where I is the identity operator and $T_\beta g(\alpha,\beta) = g(\alpha,\beta p)$, $R_\alpha g(\alpha,\beta) = g(-\alpha,\beta)$ for a function $g(\alpha,\beta)$. Thus

$$\omega T_{n',n+1}^{(e+,E-)}(\alpha,\beta) = \langle e_p(\beta V_+)E_p(\alpha V_-)V_+ f_n, f_{n'}\rangle$$
$$= \frac{1}{\beta}(I - T_\beta)R_\alpha T_{n'n}^{(e+,E-)}(\alpha,\beta) \tag{35}$$

$$(-1)^n \omega T_{n',n-1}^{(e+,E-)}(\alpha,\beta) = \frac{p}{\alpha}(T_\alpha^{-1} - I)T_{n'n}^{(e+,E-)}(\alpha,\beta). \tag{36}$$

Furthermore, induction with respect to $k+\ell$ yields $[H, \alpha^\ell \beta^k V_+^k V_-^\ell] = \frac{1}{2}(k-\ell)\alpha^\ell\beta^k V_+^k V_-^\ell$ and this implies

$$[H, e_p(\beta V_+)E_p(\alpha V_-)] = \tfrac{1}{2}(\beta\partial_\beta - \alpha\partial_\alpha)e_p(\beta V_+)E_p(\alpha V_-)$$

so

$$\frac{1}{2}(n'-n)T^{(e+,E-)}_{n'n}(\alpha,\beta)$$
$$= \langle e_p(\beta V_+)E_p(\alpha V_-)f_n, Hf_{n'}\rangle - \langle e_p(\beta V_+)E_p(\alpha V_-)Hf_n, f_{n'}\rangle$$
$$= \langle [H, e_p(\beta V_+)e_p(\alpha V_-)]f_n, f_{n'}\rangle = \tfrac{1}{2}(\beta\partial_\beta - \alpha\partial_\alpha)T^{(e+,E-)}_{n'n}(\alpha,\beta). \qquad (37)$$

Thus, the operators

$$\tilde{V}_+ = \frac{1}{\beta}(I - T_\beta)R_\alpha \qquad \tilde{V}_- = \frac{p}{\alpha}(T_\alpha^{-1} - I) \qquad \tilde{H} = \frac{1}{2}(\alpha\partial_\alpha - \beta\partial_\beta) + \frac{n'}{2}$$

and the basis functions $f_n = T^{(e+,E-)}_{n'n}$ define a two-variable realization of relations (11), hence, a realization of the representation (ω). Similar considerations apply to all of the matrix elements $T^{(e\pm,E\mp)}_{n'n}$, $T^{(E\pm,e\mp)}_{n'n}$, as well as to $T^{(e\pm,e\mp)}_{n'n}$, $T^{(E\pm,E\mp)}_{n'n}$ [16].

3. Tensor products of Euclidean q-superalgebra representations

We study the tensor product of two irreducible representations (ω_1) and (ω_2) of [0, 0] in which the coproduct is (7):

$$L = \Delta(H) = H\otimes I + I\otimes H \qquad F_\pm = \Delta(V_\pm) = V_\pm\otimes q^H + q^{-H}\otimes V_\pm.$$

The operators $F_\pm$, L satisfy the same relations as the operators $V_\pm$, H:

$$[L, F_\pm] = \pm\tfrac{1}{2}F_\pm \qquad \{F_+, F_-\} = 0. \qquad (38)$$

Each irreducible representation (ω_1), (ω_2) is defined on $L_2[0, 2\pi]$ by the prescription (15). To make sense of the operators (7) on a dense subspace of the tensor product space $L[0, 2\pi]\otimes L[0, 2\pi]$ we proceed as follows. The Hilbert space inner product is

$$\langle f, g\rangle = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f(\theta_1,\theta_2)\overline{g(\theta_1,\theta_2)}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2.$$

An orthonormal basis is $\{f_{n_1n_2} = \mathrm{e}^{i(n_1\theta_1+n_2\theta_2)}, n_j = 0, \pm1, \pm2, \ldots\}$. Using (13) we define the coproduct operators as

$$F_+ = \omega_1 z_1 T_2^{\frac{1}{2}} + \omega_2 z_2 T_1^{-\frac{1}{2}} \qquad F_- = \frac{\omega_1}{z_1}R_1T_2^{\frac{1}{2}} + \frac{\omega_2}{z_2}R_2T_1^{-\frac{1}{2}}$$
$$L = \frac{1}{2}z_1\frac{\mathrm{d}}{\mathrm{d}z_1} + \frac{1}{2}z_2\frac{\mathrm{d}}{\mathrm{d}z_2} \qquad (39)$$

where $R_j = R_{z_j}$, $T_j = T_{z_j}$, and $z_j = \mathrm{e}^{i\theta_j}$. In order to satisfy relations (38) we require that $z_1z_2 = -z_2z_1$, i.e. that the variables z_1, z_2 anticommute.

Thus, if f satisfies

$$F_+F_-f = \lambda f \qquad Lf = \frac{h}{2}f \qquad (40)$$

it should have the form $f[z,t] = t^h g_h[z]$ where

$$(\omega_1 + \omega_2 z q^{1/2}(-q^{-\frac{1}{2}})^h)\left(\omega_1(-1)^h + \frac{\omega_2(-q^{-\frac{1}{2}})^h)}{zq^{1/2}}\right)g_h[-zq] = \lambda g_h[z]$$

and $z = z_2z_1^{-1}$, $t = z_1$. Note that $zt = -tz$.

Just as in [16], there is an infinite parameter family of bases associated with the eigenvalue problem. We shall focus on those two that are the simplest in structure. The first corresponds to the basis of eigenvectors $f_{kh}^{(1)} = t^h g_h^{(1)}[z]$

$$g_h^{(1)}[z] = i^{hk} \frac{((-1)^{h+1} q^{(1-h)/2} \omega_2 z/\omega_1; -q)_\infty}{(q^{(1-h)/2} \omega_2/\omega_1 z; -q)_\infty} z^k \qquad k = 0, \pm 1, \ldots$$

with eigenvalues $\lambda_k^{(1)} = \omega_1^2 (-q)^k (-1)^h$. The second has a basis of eigenvectors

$$g_h^{(2)}[z] = i^{hk} \frac{(-q^{(1+h)/2} \omega_1 z/\omega_2; -q)_\infty}{((-1)^h q^{(1+h)/2} \omega_1/\omega_2 z; -q)_\infty} z^{h+k} \qquad k = 0, \pm 1, \ldots$$

with eigenvalues $\lambda_h^{(2)} = \omega_2^2 (-q)^k (-1)^h$. Setting $f_{kh}^{(j)}[z, t] = g_h^{(j)}[z] t^h$, $j = 1, 2$, we have

$$F_+ f_{kh}^{(j)} = \omega_j (-i q^{1/2})^k f_{k,h+1}^{(j)} \tag{41}$$

$$F_- f_{kh}^{(j)} = \omega_j (-i q^{1/2})^k (-1)^h f_{k,h-1}^{(j)} \tag{42}$$

$$L f_{kh}^{(j)} = \frac{h}{2} f_{kh}^{(j)} \qquad C f_{kh}^{(j)} = \omega_j^2 (-q)^k (-1)^h f_{kh}^{(j)}. \tag{43}$$

Thus

$$(\omega_1) \otimes_q^j (\omega_2) \equiv \sum_{k=-\infty}^{\infty} \oplus (\omega_j (iq)^{k/2})$$

i.e. in each case the tensor product decomposes into a direct sum of irreducible representations.

In a similar manner we can use the adjoint operators to (39) and compute the resolution of the tensor product representation of the factors:

$$\begin{aligned} &F'_+ = -\overline{\omega_1} z_1 R_1 T_2^{\frac{1}{2}} - \overline{\omega_2} z_2 R_2 T_1^{-\frac{1}{2}} \qquad F'_- = \frac{\overline{\omega_1}}{z_1} T_2^{\frac{1}{2}} + \frac{\overline{\omega_2}}{z_2} T_1^{-\frac{1}{2}} \\ &L' = \frac{1}{2} z_1 \frac{\mathrm{d}}{\mathrm{d}z_1} + \frac{1}{2} z_2 \frac{\mathrm{d}}{\mathrm{d}z_2}. \end{aligned} \tag{44}$$

If f' satisfies

$$F'_+ F'_- f' = \lambda' f' \qquad L' f' = \frac{h}{2} f' \tag{45}$$

it should have the form $f'[z, t] = t^h g'_h[z]$ where

$$(\overline{\omega_1} - \overline{\omega_2} z q^{1/2} (-q^{-\frac{1}{2}})^h) \left(\overline{\omega_1} (-1)^h - \frac{\overline{\omega_2} (q^{-\frac{1}{2}})^h)}{z q^{1/2}} \right) g'_h[-zq] = \lambda g'_h[z]$$

and $z = z_2 z_1^{-1}$, $t = z_1$.

Again there is an infinite parameter family of bases associated with the eigenvalue problem, but two have the simplest structure. The first corresponds to the basis of eigenvectors $f'^{(1)}_{kh} = t^h g'^{(1)}_h[z]$:

$$g'^{(1)}_h[z] = i^{hk} \frac{(q^{(1-h)/2} \overline{\omega_2} z/\overline{\omega_1}; -q)_\infty}{((-1)^{h+1} q^{(1-h)/2} \overline{\omega_2}/\overline{\omega_1} z; -q)_\infty} z^k \qquad k = 0, \pm 1, \ldots$$

with eigenvalues $\lambda'^{(1)}_k = \overline{\omega_1}^2 (-q)^k (-1)^h$. The second has a basis of eigenvectors

$$g'^{(2)}_h[z] = i^{hk} \frac{((-1)^{h+1} q^{(1+h)/2} \overline{\omega_1} z/\overline{\omega_2}; -q)_\infty}{(q^{(1+h)/2} \overline{\omega_1}/\overline{\omega_2} z; -q)_\infty} z^{h+k} \qquad k = 0, \pm 1, \ldots$$

with eigenvalues $\lambda'^{(2)}_h = \overline{\omega_2}^2(-q)^k(-1)^h$. Setting $f'^{(j)}_{kh}[z,t] = g'^{(j)}_h[z]t^h$, $j = 1, 2$, we have

$$F'_+ f'^{(j)}_{kh} = \overline{\omega_j}(iq^{1/2})^k(-1)^{h+1} f'^{(j)}_{k,h+1} \tag{46}$$

$$F'_- f'^{(j)}_{kh} = \overline{\omega_j}(iq^{1/2})^k f'^{(j)}_{k,h-1} \tag{47}$$

$$L' f'^{(j)}_{kh} = \frac{h}{2} f'^{(j)}_{kh} \qquad C f'^{(j)}_{kh} = \overline{\omega_j}^2(-q)^k(-1)^h f'^{(j)}_{kh}. \tag{48}$$

Thus

$$(\overline{\omega_1})' \otimes^j_q (\overline{\omega_2})' \equiv \sum_{k=-\infty}^{\infty} \oplus (\overline{\omega_j}(iq)^{k/2})'.$$

The functions $\{f'^{(j)}_{k'h'}\}$ form dual bases for the functions $\{f^{(j)}_{kh}\}$, $j = 1, 2$. In particular, the biorthogonality relations

$$\langle f^{(j)}_{kh}, f'^{(j)}_{k'h'} \rangle = \delta_{kk'}\delta_{hh'} \tag{49}$$

are satisfied.

The Clebsch–Gordan coefficients for the tensor product corresponding to the spectral resolution (3) are defined by

$$f^{(1)}_{kh}[z,t] = f^{(1)}_{kh}(\theta_1,\theta_2) = \sum_{n_1,n_2=-\infty}^{\infty} \begin{bmatrix} \omega_1 & \omega_2 & k \\ n_1 & n_2 & h \end{bmatrix}^{(1)} f^{\omega_1}_{n_1}(\mathrm{e}^{i\theta_1}) \otimes^1_q f^{\omega_2}_{n_2}(\mathrm{e}^{i\theta_2}). \tag{50}$$

Clearly, these coefficients vanish unless $h = n_1 + n_2$. Note that the tensor product basis is $z_1^{n_1} z_2^{n_2} = (-1)^{n_2(n_2+1)/2} t^{n_1+n_2} z^{n_2}$. There is a similar expansion for the dual basis:

$$f'^{(1)}_{kh}[z,t] = f'^{(1)}_{kh}(\theta_1,\theta_2) = \sum_{n_1,n_2=-\infty}^{\infty} \begin{bmatrix} \overline{\omega_1} & \overline{\omega_2} & k \\ n_1 & n_2 & h \end{bmatrix}'^{(1)} f^{\overline{\omega_1}}_{n_1}(\mathrm{e}^{i\theta_1}) \otimes^1_q f^{\overline{\omega_2}}_{n_2}(\mathrm{e}^{i\theta_2}). \tag{51}$$

The biorthogonality of the two bases implies the identities

$$\sum_{n_1,n_2} \begin{bmatrix} \omega_1 & \omega_2 & k \\ n_1 & n_2 & h \end{bmatrix}^{*(1)} \begin{bmatrix} \overline{\omega_1} & \overline{\omega_2} & k' \\ n_1 & n_2 & h \end{bmatrix}'^{(1)} = \delta_{kk'}$$

$$\sum_{k} \begin{bmatrix} \omega_1 & \omega_2 & k \\ n_1 & n_2 & h \end{bmatrix}^{*(1)} \begin{bmatrix} \overline{\omega_1} & \overline{\omega_2} & k \\ n'_1 & n'_2 & h \end{bmatrix}'^{(1)} = \delta_{n_1 n'_1}$$

where $n_1 + n_2 = n'_1 + n'_2 = h$ and a^* is the complex conjugate of a. Explicitly

$$\begin{bmatrix} \omega_1 & \omega_2 & k \\ n_1 & n_2 & h \end{bmatrix}^{(1)} (-1)^{n_2(n_2+1)/2}(-i)^{hk}$$

$$= \frac{((-1)^h a_k)^{n_2-k} p^{(n_2-k)(n_2-k-1)/2}}{(p;p)_{n_2-k}} {}_1\phi_1\left(\begin{matrix} 0 \\ p^{n_2-k+1} \end{matrix} ; p; (-1)^{h+1} a_k^2 p^{n_2-k} \right)$$

$$- \sum_{\ell=0}^{s_h-1} \frac{(p^\ell a_h^2; p)_\infty p^{\ell(\ell-1)/2}(-q^\ell a_h)^{k-n_2}(-1)^\ell}{(p;p)_\infty (p;p)_\ell} \tag{52}$$

where $p = -q$, $h = n_1 + n_2$, $a_h = q^{(1-h)/2}\omega_2/\omega_1$ and $s_h \geqslant 0$ is the smallest integer such that $q^{s_h} a_h < 1$. (We assume that $a_h \neq q^n$ for any integer n.) Indeed the case $s_h = 0$ of (52) follows from (50) and this result can be written as a complex contour integral. The case $s_h > 0$ can be obtained from this result by shifting the contour.

Similarly the Clebsch–Gordan coefficients for the tensor product corresponding to the first dual basis are given explicitly by (52), where a_h is now replaced by $a'_h = (-1)^{h+1}q^{(1-h)/2}\overline{\omega_2}/\overline{\omega_1}$.

With respect to the tensor product basis $\{f^{\omega_1}_{n_1} \otimes f^{\omega_2}_{n_2}\}$ the operator $e_q(\beta F_+)E_q(\alpha F_-)$ has matrix elements

$$\begin{aligned} T_{m_1m_2;n_1n_2}(\alpha,\beta) &= \langle e_p(\beta\Delta(V_+))E_p(\alpha\Delta(V_-))f^{\omega_1}_{n_1}\otimes f^{\omega_2}_{n_2}, f^{\omega_1}_{m_1}\otimes f^{\omega_2}_{m_2}\rangle \\ &= \tfrac{1}{2}\Big[T^{(e+,E-)}_{m_1n_1}\left(\alpha q^{m_2},\beta q^{m_2}\right)T^{(e+,E-)}_{m_2n_2}\left(\alpha p^{-n_1},\beta p^{-n_1}\right) \\ &\quad + T^{(e+,E-)}_{m_1n_1}\left(\alpha q^{m_2},\beta q^{m_2}\right)T^{(e+,E-)}_{m_2n_2}\left(\alpha p^{-n_1},-\beta p^{-n_1}\right) \\ &\quad + T^{(e+,E-)}_{m_1n_1}\left(-\alpha q^{m_2},\beta q^{m_2}\right)T^{(e+,E-)}_{m_2n_2}\left(\alpha p^{-n_1},\beta p^{-n_1}\right) \\ &\quad - T^{(e+,E-)}_{m_1n_1}\left(-\alpha q^{m_2},\beta q^{m_2}\right)T^{(e+,E-)}_{m_2n_2}\left(\alpha p^{-n_1},-\beta p^{-n_1}\right)\Big]. \end{aligned} \tag{53}$$

Indeed

$$\begin{aligned} e_p(\beta F_+)E_p(\alpha F_-) &= e_p(\beta V_+\otimes q^H)e_p(\beta q^{-H}\otimes V_+)E_p(\alpha V_-\otimes q^H)E_p(\alpha q^{-H}\otimes V_-) \\ &= \tfrac{1}{2}e_p(\beta V_+\otimes q^H)\Big(E_p(B)e_p(A)+E_p(B)e_p(-A)+E_p(-B)e_p(A) \\ &\quad -E_p(-B)e_p(-A)\Big)E_p(\alpha q^{-H}\otimes V_-) \end{aligned} \tag{54}$$

where $B=\alpha V_-\otimes q^H$, $A=\beta q^{-H}\otimes V_+$, $AB=-BA$, and we have used the identity

$$(-1)^{mn} = \tfrac{1}{2}(1+(-1)^m+(-1)^n-(-1)^{m+n}). \tag{55}$$

From the definition of the Clebsch–Gordan coefficients we see immediately that the identities

$$\begin{aligned} T_{m_1m_2;n_1n_2}(\alpha,\beta) &= \sum_{k=-\infty}^{\infty}\begin{bmatrix} \overline{\omega_1} & \overline{\omega_2} & k \\ n_1 & n_2 & n_1+n_2 \end{bmatrix}^{\prime(j)*} T^{(e+,E-),\omega_j(iq)^{k/2}}_{m_1+m_2,n_1+n_2}(\alpha,\beta) \\ &\quad\times\begin{bmatrix} \omega_1 & \omega_2 & k \\ m_1 & m_2 & m_1+m_2 \end{bmatrix}^{(j)} \qquad j=1,2 \end{aligned} \tag{56}$$

must hold.

4. Representations of the q-supersymmetric algebras $[v,u]$

We consider the family of algebraically irreducible representations (bounded below) $\uparrow_\lambda$, of the q-superalgebra $[-u,t]$, where $t>0, uq^{-4\lambda}, -uq^{-4\lambda}$, defined as follows [4]. A convenient orthonormal basis for the representation space is $\{e_n : n=0,1,\ldots\}$ where

$$\begin{aligned} V_-e_n &= \left[\frac{(1-p^n)(tq^{2\lambda-n+1}+u(-1)^nq^{-2\lambda})}{1+q}\right]^{\frac{1}{2}}e_{n-1} \\ V_+e_n &= -\left[\frac{(1-p^{n+1})(tq^{2\lambda-n}-u(-1)^nq^{-2\lambda})}{1+q}\right]^{\frac{1}{2}}e_{n+1} \\ He_n &= (-\lambda+\tfrac{1}{2}n)e_n. \end{aligned} \tag{57}$$

We have $V_+ = -(V_-)^*$ and $H^* = H$. The 'Casimir' operator (5) has the action

$$Ce_n = (-1)^n \mu e_n \qquad \mu = \frac{-uq^{2\lambda}}{1+q} + \frac{tq^{-2\lambda}}{1+q^{-1}}. \tag{58}$$

Note that $C^2 = \mu^2 I$ is a multiple of the identity operator.

In analogy with [4], we define the following q-analogue of matrix elements of $\uparrow_\lambda$:

$$e_{-q}(\beta V_-)E_{-q}(\alpha V_+) f_n = \sum_{n'=0}^{\infty} T_{n'n}^{(e-,E+)}(\alpha,\beta) f_{n'} \qquad |q^{2\lambda-n}\alpha\beta t/(1+q)| < 1 \tag{59}$$

where $0 < q < 1$ and $\alpha, \beta \in \mathbb{C}$. Again the motivation for the choice of base $-q$ comes from the relations

$$(V_\pm \otimes q^H)(q^{-H} \otimes V_\pm) = -q^{\pm 1}(q^{-H} \otimes V_\pm)(V_\pm \otimes q^H)$$

relevant for the coproduct. Subject only to the conditions on (59), the left-hand side of this expression is an analytic function of z, convergent in a neighbourhood of $z = 0$. Note that

$$\langle T e_n, e_{n'} \rangle = T_{n'n}. \tag{60}$$

Since the 'group' operators that we are considering are unbounded, it is not clear that relation (60) makes more than formal sense. However, by truncating the operator power series expressions for $e_p(\alpha V_-)$, $E_p(\beta V_+)$ at N terms, we see that (60) holds rigorously for the truncated series. The final result follows in the limit as $N \to \infty$.

Using the model in [4] to compute the matrix elements (which are model independent) we obtain the explicit result

$$\begin{aligned} T_{n'n}^{(e-,E+)}(\alpha,\beta) &= (-1)^{(n'-n)(n'-n+1)/2} \sqrt{\frac{(p;p)_{n'}((u/t)q^{-4\lambda};p)_n}{(p;p)_n((u/t)q^{-4\lambda};p)_{n'}}} \\ &\quad \times \left(\sqrt{\frac{t}{1+q}}\alpha\right)^{n'-n} q^{(n'-n)(7\lambda+n'-3n-1)/4} \, \frac{(\frac{(-1)^n u}{t} q^{n-4\lambda};p)_{n'-n}}{(p;p)_{n'-n}} \\ &\quad \times {}_2\phi_1 \left(\begin{matrix} p^{n'+1}, & \frac{u}{t}(-1)^n q^{n'-4\lambda} \\ p^{n'-n+1} & \end{matrix} ; p; \frac{(-1)^{n-n'+1}\alpha\beta t}{q^{n-2\lambda}(1+q)} \right) \end{aligned} \tag{61}$$

where $|q^{2\lambda-n}\alpha\beta t/(1+q)| < 1$.

These results make sense for $n > n'$ as well as $n' \geqslant n$. Indeed, for $k > 0$ we have

$$\lim_{m\to -k} \frac{1}{(p;p)_m} {}_2\phi_1 \left(\begin{matrix} A & B \\ p^{m+1} & \end{matrix} ; p, C \right) = \frac{(A,B;p)_k C^k}{(p;p)_k} {}_2\phi_1 \left(\begin{matrix} Ap^k & Bp^k \\ p^{k+1} & \end{matrix} ; p, C \right). \tag{62}$$

We also consider a second family of algebraically irreducible representations (bounded above) $\downarrow_\xi$, of the q-superalgebra $[v, u]$, where $v > 0, uq^{4\xi-1}, -uq^{4\xi}$, defined as follows. A convenient orthonormal basis for the representation space is $\{j_m : m = 0, 1, \ldots\}$ where

$$\begin{aligned} V_- j_m &= (-1)^{m+1} \left[\frac{(1-p^{m+1})(vq^{-2\xi-m} + u(-1)^m q^{2\xi})}{1+q} \right]^{\frac{1}{2}} j_{m+1} \\ V_+ j_m &= (-1)^{m+1} \left[\frac{(1-p^m)(vq^{-2\xi-m+1} - u(-1)^m q^{2\xi})}{1+q} \right]^{\frac{1}{2}} j_{m-1} \\ H j_m &= (-\xi - \tfrac{1}{2}m) j_m. \end{aligned} \tag{63}$$

We have $V_+ = -(V_-)^*$ and $H^* = H$.

For this representation we consider the matrix elements

$$(e-, E+):\quad e_p(\beta V_-)E_p(\alpha V_+)j_m = \sum_{m'} S^{(\xi)}_{m'm}(\alpha,\beta)j_{m'}.$$

$$\begin{aligned}S^{(\xi)}_{m'm}(\alpha,\beta) = \beta^{m'-m}q^{(m'-m)(-4\xi-m'-m+1)/4}\,&\frac{((-u/v)p^m q^{4\xi};\,p)_{m'-m}}{(p;\,p)_{m'-m}}\\ &\times(-1)^{(m'-m)(m'+m+1)/2}\sqrt{\left(\frac{v}{1+q}\right)^{m'-m}\frac{(p;\,p)_{m'}(-(u/v)q^{4\xi};\,p)_m}{(p;\,p)_m(-(u/v)q^{4\xi};\,p)_{m'}}}\\ &\times\left[\frac{1-i}{2}\,{}_2\phi_1\left(\begin{matrix}p^{-m}, & (v/u)q^{-4\xi+1}p^{-m}\\ p^{m'-m+1}\end{matrix};\,p;\,\frac{-\alpha\beta i u}{q^{-2\xi-m}(1+q)}\right)\right.\\ &\left.+\frac{1+i}{2}\,{}_2\phi_1\left(\begin{matrix}p^{-m}, & (v/u)q^{-4\xi+1}p^{-m}\\ p^{m'-m+1}\end{matrix};\,p;\,\frac{\alpha\beta i u}{q^{-2\xi-m}(1+q)}\right)\right].\end{aligned} \tag{64}$$

Now we form the tensor product representation

$$\downarrow_\xi [v,u]\otimes\uparrow_\lambda[-u,t]\cong[v,t] \tag{65}$$

of $[v,t]$. Rather than define the operators determining the tensor product in the form (7), we will use the equivalent definition

$$F_\pm = V_\pm\otimes q^H + (-1)^{2H'}q^{-H}\otimes V_\pm \qquad L = H\otimes I + I\otimes H \tag{66}$$

where now

$$(A\otimes B)j_m\otimes e_n = Aj_m\otimes Be_n \qquad 2H'j_m = (-1)^m j_m. \tag{67}$$

With this definition the variables commute in our functional models. The invariant operator is

$$C = F_+F_- + \frac{v}{1+q}q^{2L} + \frac{t}{1+q^{-1}}q^{-2L}.$$

To decompose this representation we compute the common eigenfunctions of L and C. Clearly, eigenfunctions of L with eigenvalue $-\lambda-\xi+\alpha$, $\alpha\geqslant 0$ are just those linear combinations of the basis vectors $J^\alpha_m = j_m\otimes e_n$ where $n = 2\alpha+m$, $m = 0,1,\ldots$.. For $\alpha < 0$, they are linear combinations of the basis vectors $J^\alpha_m = j_m\otimes e_n$ where $n = 2\alpha+m$, $n = 0,1,\ldots$. Taking the case $\alpha\geqslant 0$ and applying C to the ON set $\{J^\alpha_m\}$ we find

$$\begin{aligned}-\frac{(1+q)}{\sqrt{tvq}}CJ^\alpha_m = &\left[(1-p^{m+1})(1-p^{2\alpha+m+1})\left(1+\frac{u}{v}p^m q^{4\xi}\right)\right.\\ &\left.\times\left(1-\frac{u}{t}(-1)^{2\alpha}q^{-4\lambda+2\alpha}p^m\right)\right]^{\frac12}J^\alpha_{m+1}\\ &+\left[(1-p^m)(1-p^{2\alpha+m})\left(1+\frac{u}{v}p^{m-1}q^{4\xi}\right)\left(1+\frac{u}{t}p^{m+2\alpha}q^{-4\lambda-1}\right)\right]^{\frac12}J^\alpha_{m-1}\\ &-\frac{1}{\sqrt{vt}}\Big\{tq^{2\lambda+2\xi-2\alpha-\frac12}+vq^{-2\xi-2\lambda+2\alpha-\frac12}\\ &-q^{-2\lambda+2\alpha-\frac12}(1-p^{m+1})(vq^{-2\xi}+up^m q^{2\xi})\\ &-q^{2\xi-\frac12}(1-p^{m+2\alpha})(tq^{2\lambda-2\alpha}+u(-1)^{2\alpha}p^m q^{-2\lambda})\Big\}J^\alpha_m.\end{aligned} \tag{68}$$

The operator C is self-adjoint. If we introduce the spectral transform of this operator so that C corresponds to multiplication by the transform variable x, then (68) takes the form of a three-term recurrence relation for orthogonal polynomials $J_m^\alpha(x)$ of order m in x. Indeed, comparing (68) with the three-term recurrence relation for the continuous Askey–Wilson polynomials [6, page 173]

$$p_m(x) \equiv p_m(x; a, b, c, d|Q)$$
$$= (ab, ac, ad; Q)_m a^{-m} {}_4\phi_3 \left(\begin{matrix} Q^{-m}, & abcdQ^{m-1}, & ae^{i\theta}, & ae^{-i\theta} \\ ab, & ac, & ad & \end{matrix} ; Q, Q \right) \qquad (69)$$

we get a match with $Q = p$

$$a = \sqrt{\frac{vq}{t}} q^{-2\lambda-2\xi+2\alpha} \qquad b = -\sqrt{\frac{tq}{v}} q^{2\lambda+2\xi}(-1)^{2\alpha} \qquad c = \frac{u}{\sqrt{vtq}} q^{-2\lambda+2\xi}(-1)^{2\alpha}$$

$d = 0$, and $C \sim -2x\sqrt{vtq}/(1+q)$. Making the identification

$$J_m^\alpha(x, s) = \frac{\left[\frac{(p^{m+1}, abp^m, acp^m, bcp^m; p)_\infty}{2\pi}\right]^{\frac{1}{2}} p_m(x; a, b, c, 0) s^{2\alpha}}{\left(\sqrt{\frac{v}{t}} q^{-2\lambda-2\xi+2\alpha+\frac{1}{2}} e^{i\theta}, -\sqrt{\frac{t}{v}} q^{2\lambda+2\xi+\frac{1}{2}}(-1)^{2\alpha} e^{i\theta}, \frac{u}{\sqrt{vt}} q^{-2\lambda+2\xi-\frac{1}{2}}(-1)^{2\alpha} e^{i\theta}; p\right)_\infty}$$
$$= K_m^\alpha(x) s^{2\alpha} \qquad (70)$$

where $s = e^{i\phi}$ and $x = \cos\theta$, we can verify that (68) holds, as well as the orthogonality relations

$$\langle J_m^\alpha, J_{m'}^{\alpha'}\rangle = \delta_{m,m'}\delta_{\alpha,\alpha'}$$
$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-1}^{1} \rho(x)\, dx \int_0^{2\pi} d\phi\; f(x, s)\overline{g(x, s)} \qquad (71)$$
$$\rho(x) = (1-x^2)^{-\frac{1}{2}} \left(e^{i\theta}, e^{-i\theta}, -e^{i\theta}, -e^{-i\theta}, p^{\frac{1}{2}} e^{i\theta}, p^{\frac{1}{2}} e^{-i\theta}, -p^{\frac{1}{2}} e^{i\theta}, -p^{\frac{1}{2}} e^{-i\theta}; p\right)_\infty.$$

For $\alpha = -\beta$, $\beta = 0, 1, \ldots$, the expression for $J_m^\alpha(x, s)$ can be obtained by analytic continuation and a standard limiting procedure, analogous to (62).

It is straightforward, though tedious, to verify that in terms of the new variables $r = e^{i\theta}$, s the action of the operators $F_\pm$, L is

$$F_+ = \frac{s}{\sqrt{1+q}} \left(-\sqrt{t}q^{\lambda+\xi} T_s^{-1/2} - \sqrt{v} r q^{-\xi-\lambda+\frac{1}{2}} T_s^{1/2}\right) R_r$$
$$F_- = \frac{1}{s\sqrt{1+q}} \left(\sqrt{t}q^{\lambda+\xi+\frac{1}{2}} T_s^{-1/2} - \frac{\sqrt{v}}{r} q^{\xi-\lambda} T_s^{1/2}\right) R_r \qquad (72)$$
$$L = -\xi - \lambda + \frac{1}{2} s\frac{\partial}{\partial s} \qquad C = -\frac{\sqrt{vtq}}{1+q}\left(r + \frac{1}{r}\right).$$

Here, $R_r g(r) = g(-r)$.

Applying these operators to the basis $f_h^{\pm x_0}(x, s) = \left(s^h/\sqrt{\rho(x)}\right)\delta(x \mp x_0)$, we see that the algebra action becomes

$$F_+ f_h^{\pm x_0} = \frac{1}{\sqrt{1+q}} \left(-\sqrt{t} q^{\lambda+\xi-h/2} - \sqrt{v} r q^{-\xi-\lambda+(h+1)/2}\right) f_{h+1}^{\mp x_0}$$
$$F_- f_h^{\pm x_0} = \frac{1}{\sqrt{1+q}} \left(\sqrt{t} q^{\lambda+\xi+(1-h)/2} - \frac{\sqrt{v}}{r} q^{\xi-\lambda+h/2}\right) f_{h-1}^{\mp x_0} \qquad (73)$$
$$L f_h^{\pm x_0} = \left(-\xi - \lambda + \frac{h}{2}\right) f_h^{\pm x_0} \qquad C = \mp 2x_0 \frac{\sqrt{vtq}}{1+q}$$

for $x_0 \geqslant 0$. Thus the algebra action preserves $|x|$ and decomposes the representation space for fixed $|x| = x_0$ into the direct sum of two irreducible representations. The basis vector f_h^x belongs to the space of the irreducible representation $(\lambda+\xi, |x|)$ if $(-1)^h x > 0$, and to the space of the irreducible representation $(\lambda+\xi, -|x|)$ if $(-1)^h x < 0$. (We ignore the measure 0 case $x = 0$).

For the reduced tensor product we consider the matrix elements

$$(e-, E+): \quad e_p(\beta F_-)E_p(\alpha F_+)f_h^x = \sum_{h'} U_{h'h}^{(-1)^{h+h'}x}(\alpha,\beta) f_{h'}^{(-1)^{h+h'}x} \tag{74}$$

$$\begin{aligned} U_{h'h}^x(\alpha,\beta) = {} & \left(\frac{\beta\sqrt{v}}{r\sqrt{1+q}}\right)^{h-h'} q^{(h-h')(h+h'+1)/4} \frac{(-1)^{(h-h')(h-h'+1)/2}}{q^{\mu(h-h')}(p;p)_{h-h'}} \\ & \times \left((-1)^{1-h'}\sqrt{\frac{t}{v}}\, r q^{2\mu+\frac{1}{2}} p^{-h}; p\right)_{h-h'} \\ & \times \left[\frac{1-i}{2}\, {}_2\phi_1\left(\begin{matrix} \frac{(-1)^{h'}}{r}\sqrt{\frac{vq}{t}} q^{-2\mu} p^h, & (-1)^{h'}\sqrt{\frac{vq}{t}} r q^{-2\mu} p^h \\ & p^{h-h'+1} \end{matrix}; p; \frac{-\alpha\beta q^{2\mu} i t}{q^h(1+q)}\right)\right. \\ & \left. + \frac{1+i}{2}\, {}_2\phi_1\left(\begin{matrix} \frac{(-1)^{h'}}{r}\sqrt{\frac{vq}{t}} q^{-2\mu} p^h, & (-1)^{h'}\sqrt{\frac{vq}{t}} r q^{-2\mu} p^h \\ & p^{h-h'+1} \end{matrix}; p; \frac{\alpha\beta q^{2\mu} i t}{q^h(1+q)}\right)\right] \end{aligned} \tag{75}$$

where $\mu = \lambda + \xi$ and $x = (r + r^{-1})/2$.

Comparing (66), (71), (73), we have proved the direct integral decomposition

$$\downarrow_\xi [v,u] \otimes \uparrow_\lambda [-u,t] \cong \int_0^1 \rho(x)\,\{(\lambda+\xi, x) \oplus (\lambda+\xi, -x)\}\,\mathrm{d}x. \tag{76}$$

The functions $J_m^{h/2}(x,s)$ are, essentially, the Clebsch–Gordan coefficients for this decomposition; the orthogonality and completeness relations for the corresponding Askey–Wilson polynomials are the unitarity conditions for the CG coefficients.

The decomposition (76) can be used to obtain an identity relating the matrix elements (61), (64) and (75). We can compute the matrix element

$$T_{m'n',mn}(\alpha,\beta) = \langle e_p(\beta F_-)E_p(\alpha F_+) j_m \otimes e_n,\, j_{m'} \otimes e_{n'}\rangle$$

in two different ways. On one hand we have the integral representation

$$\begin{aligned} T_{m'n',mn}(\alpha,\beta) &= \langle e_p(\beta F_-)E_p(\alpha F_+) J_m^{h/2},\, J_{m'}^{h'/2}\rangle \\ &= \langle U_{h'h}^{(\cdot)}(\alpha,\beta) K_m^{h/2}((-1)^{h+h'}\cdot),\, K_{m'}^{h'/2}(\cdot)\rangle \end{aligned} \tag{77}$$

where $J_m^{h/2}(x,s) = K_m^{h/2}(x)s^h$, and $n = h+m, n' = h'+m'$. On the other hand, from relations (7), (22) and (55) we have the formal identity

$$\begin{aligned} e_p(\beta F_-)E_p(\alpha F_+) &= e_p(\beta V_- \otimes q^H) e_p(\beta q^{-H} \otimes V_-) E_p(\alpha V_+ \otimes q^H) E_p(\alpha q^{-H} \otimes V_+) \\ &= \tfrac{1}{2} e_p(\beta V_- \otimes q^H)\Big(E_p(B)e_p(A) + E_p(B)e_p(-A) \\ &\quad + E_p(-B)e_p(A) - E_p(-B)e_p(-A)\Big) E_p(\alpha q^{-H} \otimes V_+) \end{aligned} \tag{78}$$

where $B = \alpha V_+ \otimes q^H$, $A = \beta q^{-H} \otimes V_-$, $AB = -BA$. Thus

$$\begin{aligned}2T_{m'n',mn}(\alpha,\beta) &= S^{(\xi)}_{m'm}(\alpha q^{-\lambda+n'/2}, \beta q^{-\lambda+n'/2}) T^{(e-,E+),\lambda}_{n'n}(\alpha(-1)^m q^{\xi+m/2}, \beta(-1)^m q^{\xi+m/2}) \\ &+ S^{(\xi)}_{m'm}(\alpha q^{-\lambda+n'/2}, \beta q^{-\lambda+n'/2}) T^{(e-,E+),\lambda}_{n'n}(\alpha(-1)^m q^{\xi+m/2}, -\beta(-1)^m q^{\xi+m/2}) \\ &+ S^{(\xi)}_{m'm}(-\alpha q^{-\lambda+n'/2}, \beta q^{-\lambda+n'/2}) T^{(e-,E+),\lambda}_{n'n}(\alpha(-1)^m q^{\xi+m/2}, \beta(-1)^m q^{\xi+m/2}) \\ &- S^{(\xi)}_{m'm}(-\alpha q^{-\lambda+n'/2}, \beta q^{-\lambda+n'/2}) T^{(e-,E+),\lambda}_{n'n}(\alpha(-1)^m q^{\xi+m/2}, -\beta(-1)^m q^{\xi+m/2})\end{aligned} \tag{79}$$

where the functions $T_{n'm',nm}(\alpha,\beta)$ are the matrix elements of the operator $e_p(\beta F_-)E_p(\alpha F_+)$ in the tensor product basis $\{j_m \otimes e_n\}$.

Next we consider a degenerate case of the tensor product with some special features, a representation of the q-supereuclidean algebra $[0,0]$ of the form $\downarrow'_\xi [0,u] \otimes \uparrow'_\lambda [-u,0]$. Here the family of algebraically irreducible representations (bounded below) $\uparrow'_\lambda$, of the q-superalgebra $[-u,0]$, where $u>0$, is defined as follows. An orthogonal basis for the representation space is $\{e_n : n = 0,1,\ldots\}$ where

$$\begin{aligned} V_- e_n &= \left[\frac{(1-p^n)uq^{-2\lambda}}{1+q}\right]^{\frac12} e_{n-1} \qquad H e_n = \left(-\lambda + \frac{n}{2}\right) e_n \\ V_+ e_n &= \left[\frac{(1-p^{n+1})uq^{-2\lambda}}{1+q}\right]^{\frac12} (-1)^n e_{n+1}. \end{aligned} \tag{80}$$

The representation space is a Hilbert space with indefinite inner product. Indeed

$$\langle e_n, e_{n'}\rangle = \delta_{nn'}(-1)^{n(n-1)/2} \qquad n, n' = 0, 1, \ldots \tag{81}$$

and $V_+ = (V_-)^*$, $H^* = H$. We define $H' = \lambda I + H$.

The following matrix elements of $\uparrow'_\lambda$:

$$e_{-q}(\beta V_-)E_{-q}(\alpha V_+)e_n = \sum_{n'=0}^{\infty} T'^{(e-,E+)}_{n'n}(\alpha,\beta)e_{n'} \tag{82}$$

are given by

$$\begin{aligned} T'^{(e-,E+)}_{n'n}(\alpha,\beta) &= q^{\frac12(n'-n)(n'+n-1)}\alpha^{n'-n}\left(\frac{uq^{-2\lambda}}{1+q}\right)^{\frac12(n'-n)} \frac{\sqrt{(p^{n+1};p)_{n'-n}}}{(p;p)_{n'-n}} \\ &\times \left[\frac{1-i}{2}\,{}_1\phi_1\left(\begin{matrix} p^{n'+1} \\ p^{n'-n+1}\end{matrix} ; p; \frac{-iq^{n'}\alpha\beta u}{p^n q^{2\lambda}(1+q)}\right)\right. \\ &+ \left.\frac{1+i}{2}\,{}_1\phi_1\left(\begin{matrix} p^{n'+1} \\ p^{n'-n+1}\end{matrix} ; p; \frac{iq^{n'}\alpha\beta u}{p^n q^{2\lambda}(1+q)}\right)\right]. \end{aligned} \tag{83}$$

Here

$$\langle T'e_n, e_{n'}\rangle = T'_{n'n}(-1)^{n(n-1)/2}. \tag{84}$$

We also consider a second family of algebraically irreducible representations (bounded above) $\downarrow'_\xi$, of the q-superalgebra $[0,u]$, where $u>0$, defined as follows. A convenient orthogonal basis for the representation space with indefinite inner product is $\{j_m : m = 0, 1, \ldots\}$ where

$$\langle j_m, j_{m'}\rangle = \delta_{mm'}(-1)^{m(m+1)/2} \qquad m, m' = 0, 1, \ldots \tag{85}$$

and

$$V_- j_m = \left[\frac{(1-p^{m+1})uq^{2\xi}}{1+q}\right]^{\frac{1}{2}} j_{m+1} \qquad H j_m = -\left(\xi + \frac{m}{2}\right) j_m$$
$$V_+ j_m = \left[\frac{(1-p^{m})uq^{2\xi}}{1+q}\right]^{\frac{1}{2}} j_{m-1}. \tag{86}$$

We have $V_+ = (V_-)^*$ and $H^* = H$. We define $H' = \xi I + H$.

For this representation we consider the matrix elements

$$(e-, E+): \quad e_p(\beta V_-)E_p(\alpha V_+) j_m = \sum_{m'} S'^{(\xi)}_{m'm}(\alpha,\beta) j_{m'}$$

$$S'^{(\xi)}_{m'm}(\alpha,\beta) = (\beta)^{m'-m}\left(\frac{uq^{2\xi}}{1+q}\right)^{\frac{1}{2}(m'-m)} \frac{\sqrt{(p^{m+1};p)_{m'-m}}}{(p;p)_{m'-m}}$$
$$\times {}_2\phi_1\left(\begin{matrix} p^{-m}, & 0 \\ p^{m'-m+1} & \end{matrix}; p; \frac{-\alpha\beta p^m u}{q^{-2\xi}(1+q)}\right). \tag{87}$$

Now we form the tensor product representation

$$\downarrow'_\xi [0,u] \otimes \uparrow'_\lambda [-u,0] \cong [0,0] \tag{88}$$

of $[0,0]$ using the definition (66). The invariant operator is $C = F_+F_-$. Note that $F_+^* = F_-$, $L^* = L$ and that a basis for the indefinite inner product space is $\{j_m \otimes e_n,\ m,n = 0,1,\ldots\}$ where

$$\langle j_m \otimes e_n, j_{m'} \otimes e_{n'}\rangle = \langle j_m, j_{m'}\rangle\langle e_n, e_{n'}\rangle = \delta_{mm'}\delta_{nn'}(-1)^{\frac{1}{2}(m+n)(m-n+1)}. \tag{89}$$

To decompose this representation we compute the common eigenfunctions of L and C. Clearly, eigenfunctions of L with eigenvalue $-\lambda - \xi + \alpha$, $\alpha \geqslant 0$ are just those linear combinations of the basis vectors $J_m^\alpha = j_m \otimes e_n$ where $n = 2\alpha + m,\ m = 0,1,\ldots$. For $\alpha < 0$, they are linear combinations of the basis vectors $J_m^\alpha = j_m \otimes e_n$ where $n = 2\alpha + m,\ n = 0,1,\ldots$. Note that

$$\|J_m^\alpha\|^2 \equiv \langle J_m^\alpha, J_m^\alpha\rangle = \frac{(-1)^{[\alpha/2]}}{2}\left[1 - (-1)^\alpha + (-1)^m + (-1)^{\alpha+m}\right]. \tag{90}$$

Taking the case $\alpha \geqslant 0$ and applying C to the orthogonal set $\{J_m^\alpha\}$ we find

$$\frac{(1+q)q^{2(\lambda-\xi)}}{u} C J_m^\alpha = (-1)^\alpha p^m q^{\alpha/2+\frac{1}{2}}\left[(1-p^{m+1})(1-p^{\alpha+m+1})\right]^{\frac{1}{2}} J_{m+1}^\alpha$$
$$+ p^m q^{\alpha/2-\frac{1}{2}}\left[(1-p^m)(1-p^{\alpha+m})\right]^{\frac{1}{2}} J_{m-1}^\alpha$$
$$- p^m\left[q^\alpha(i - p^{m+1}) + (-1)^\alpha(1-p^{\alpha+m})\right] J_m^\alpha. \tag{91}$$

The operator C is self-adjoint. If we introduce the spectral transform of this operator so that C corresponds to multiplication by the transform variable x, then (91) takes the form of a three-term recurrence relation for orthogonal polynomials $J_m^\alpha(x)$ of order m in x. Indeed, comparing (91) with the three-term recurrence relation for the continuous Askey–Wilson polynomials we see that we get a match for a special case of the little q-Jacobi polynomials

$$p_n(x;a,b;Q) = {}_2\phi_1\left(\begin{matrix} Q^{-n}, & abQ^{n+1} \\ aQ & \end{matrix}; Q, xQ\right) \tag{92}$$

where $Q = p$, $a = p^\alpha$ and $C \sim xuq^{2(\xi-\lambda)}(-1)^{\alpha+1}/(1+q)$. Making the identification

$$J_m^\alpha(x,s) = \left[\frac{q^{-m(\alpha+1)}(p^{\alpha+1};p)_\infty(p^{\alpha+1};p)_m}{(p;p)_m}\right]^{\frac{1}{2}} p_m(x;p^\alpha,0;p)s^\alpha \tag{93}$$

we have

$$\langle J_m^\alpha, J_m^{\alpha'}\rangle = \delta_{\alpha\alpha'}\frac{(-1)^{[\alpha/2]}}{2}\left[1-(-1)^\alpha+(-1)^m+(-1)^{\alpha+m}\right] \tag{94}$$

where

$$\langle f(x)s^\alpha, g(x)s^{\alpha'}\rangle \equiv \delta_{\alpha\alpha'}(-1)^{\alpha(\alpha-1)/2}\sum_{y=0}^{\infty}\frac{p^{y(\alpha+1)}}{(p;p)_y}f(p^y)g(p^y). \tag{95}$$

For fixed odd integer α, this defines a true inner product for little q-Jacobi polynomials; for even α it is just a bilinear product since the discrete measure can be negative. It is not dificult to verify that on this basis the superalgebra acts as follows:

$$\begin{aligned} &F_+ = q^{\xi-\lambda}\sqrt{\frac{u}{q+1}}sR_s \qquad F_- = q^{\xi-\lambda}\sqrt{\frac{u}{q+1}}\frac{x}{s} \\ &L = -(\xi+\lambda)+\tfrac{1}{2}s\partial_s \end{aligned} \tag{96}$$

where $R_sG(s) = G(-s)$. For fixed x this representation is equivalent to the representation $(\omega) = \left(q^{\xi-\lambda}\sqrt{xu/(q+1)}\right)$ of $[0,0]$, as studied in section 2. Thus we have derived the decomposition

$$\downarrow'_\xi [0,u]\otimes\uparrow'_\lambda [-u,0] \cong \oplus\sum_{y=0}^{\infty}\left(q^{\xi-\lambda}\sqrt{\frac{p^y u}{q+1}}\right)[0,0]. \tag{97}$$

Again, the functions $J_m^h(x,s)$ are essentially the Clebsch–Gordan coefficients for this decomposition; the orthogonality and completeness relations for the corresponding little p-Jacobi polynomials are the unitarity conditions for the CG coefficients. Also, the decomposition (97) can be used to obtain an identity relating the matrix elements (83), (87) and the adjoint of (30). We can compute the matrix element

$$T'_{m'n',mn}(\alpha,\beta) = \langle e_p(\beta F_-)E_p(\alpha F_+)j_m\otimes e_n, j_{m'}\otimes e_{n'}\rangle$$

in two different ways. On one hand we have the representation

$$\begin{aligned} T'_{m'n',mn}(\alpha,\beta) &= \langle e_p(\beta F_-)E_p(\alpha F_+)J_m^h, J_{m'}^{h'}\rangle \\ &= \langle U'^{(\cdot)}_{h'h}(\alpha,\beta)K_m^h(\cdot), K_{m'}^{h'}(\cdot)\rangle \end{aligned} \tag{98}$$

where $J_m^h(x,s) = K_m^h(x)s^h$, $U'^{(\omega)}_{h'h}(\alpha,\beta) = x^{(h-h')/2}L'_{h'h}(\alpha,\beta)$, $n = h+m$, $n' = h'+m'$ and the L' are essentially the adoint matrix elements to (30):

$$\begin{aligned} L'^{(x)}_{h'h}(\alpha,\beta) = \frac{\left(\beta q^{\xi-\lambda}\sqrt{ux/(1+q)}\right)^{h-h'}}{(p;p)_{h-h'}}&\left[{}_1\phi_1\left(\begin{matrix}0\\ p^{h-h'+1}\end{matrix};p,\frac{i\alpha\beta ux(-1)^{h+1}}{q^{2\lambda-2\xi}(1+q)}\right)\frac{1-i}{2}\right.\\ &\left.+\,{}_1\phi_1\left(\begin{matrix}0\\ p^{h-h'+1}\end{matrix};p,(-1)^h\frac{i\alpha\beta q^{2\xi-2\lambda}ux}{1+q}\right)\frac{1+i}{2}\right]. \end{aligned} \tag{99}$$

On the other hand

$$
\begin{aligned}
2T'_{m'n',mn}(\alpha,\beta) &= S'^{(\xi)}_{m'm}(\alpha q^{-\lambda+n'/2},\beta q^{-\lambda+n'/2})T'^{(e-,E+),\lambda}_{n'n}(\alpha(-1)^m q^{\xi+m/2},\beta(-1)^m q^{\xi+m/2})\\
&\quad + S'^{(\xi)}_{m'm}(\alpha q^{-\lambda+n'/2},\beta q^{-\lambda+n'/2})T'^{(e-,E+),\lambda}_{n'n}(\alpha(-1)^m q^{\xi+m/2},-\beta(-1)^m q^{\xi+m/2})\\
&\quad + S'^{(\xi)}_{m'm}(-\alpha q^{-\lambda+n'/2},\beta q^{-\lambda+n'/2})T'^{(e-,E+),\lambda}_{n'n}(\alpha(-1)^m q^{\xi+m/2},\beta(-1)^m q^{\xi+m/2})\\
&\quad - S'^{(\xi)}_{m'm}(-\alpha q^{-\lambda+n'/2},\beta q^{-\lambda+n'/2})\\
&\quad \times T'^{(e-,E+),\lambda}_{n'n}(\alpha(-1)^m q^{\xi+m/2},-\beta(-1)^m q^{\xi+m/2}).
\end{aligned}
\tag{100}
$$

For our last example we consider another degenerate case of the tensor product with some special features, a representation of the q-superoscillator algebra $[0,-u]$ of the form $\omega'[0,0]\otimes\uparrow''_\lambda[0,-u]$. Here the family of algebraically irreducible representations (bounded below) $\uparrow''_\lambda$, of the q-superalgebra $[0,-u]$, where $u>0$, is defined as follows. An orthonormal basis for the representation space is $\{e_n : n=0,1,\ldots\}$ where

$$
\begin{aligned}
V_-e_n &= \left[\frac{(1-p^n)uq^{2\lambda-n+1}}{1+q}\right]^{\frac12} e_{n-1} \qquad He_n = \left(-\lambda+\frac{n}{2}\right)e_n\\
V_+e_n &= \left[\frac{(1-p^{n+1})uq^{2\lambda-n}}{1+q}\right]^{\frac12} e_{n+1}.
\end{aligned}
\tag{101}
$$

This Hilbert space representation satisfies $V_+=(V_-)^*$, $H^*=H$. We define $H'=\lambda I+H$.

The following matrix elements of $\uparrow''_\lambda$:

$$
e_{-q}(\beta V_-)E_{-q}(\alpha V_+)e_n = \sum_{n'=0}^{\infty} T''^{(e-,E+)}_{n'n}(\alpha,\beta)e_{n'}
\tag{102}
$$

are given by

$$
\begin{aligned}
T''^{(e-,E+)}_{n'n}(\alpha,\beta) &= (-1)^{(n'-n)(n'-n+1)/2}\sqrt{\frac{(p;p)_{n'}}{(p;p)_n}}\left(-\sqrt{\frac{u}{1+q}}\alpha\right)^{n'-n}\frac{q^{(n'-n)(7\lambda+n'-3n-1)/4}}{(p;p)_{n'-n}}\\
&\quad\times{}_2\phi_1\left(\begin{matrix}p^{n'+1}, & 0\\ p^{n'-n+1} & \end{matrix};p;\frac{(-1)^{n-n'}\alpha\beta u}{q^{n-2\lambda}(1+q)}\right)
\end{aligned}
\tag{103}
$$

where $|q^{2\lambda-n}\alpha\beta u/(1+q)|<1$.

The family of algebraically irreducible representations ω', of the q-superalgebra $[0,0]$, is defined by the action (12) on the basis $\{f_m\}$, except that here we have an indefinite inner product

$$
\langle f_m, f_{m'}\rangle = \delta_{mm'}(-1)^{m(m+1)/2} \qquad m,m'=0,\pm1,\ldots.
\tag{104}
$$

Thus $V_+=(V_-)^*$ and $H^*=H$. We define $H'=H$. The matrix elements are given by (29).

Now we form the tensor product representation

$$
\omega'[0,0]\otimes\uparrow''_\lambda[0,-u]\cong[0,-u]
\tag{105}
$$

of $[0,-u]$ using definition (66). The invariant operator is $C=F_+F_- - uq^{-2L}/(1+q^{-1})$. Note that $F_+^*=F_-$, $L^*=L$ and that a basis for the indefinite inner product space is $\{f_m\otimes e_n,\ n,\pm m=0,1,\ldots\}$ where

$$
\langle f_m\otimes e_n, f_{m'}\otimes e_{n'}\rangle = \langle f_m,f_{m'}\rangle\langle e_n,e_{n'}\rangle = \delta_{mm'}\delta_{nn'}(-1)^{\frac12 m(m+1)}.
\tag{106}
$$

Again, to decompose this representation we compute the common eigenfunctions of L and C. Clearly, eigenfunctions of L with eigenvalue $-\lambda+\alpha/2$ are just those linear combinations of the basis vectors $J_n^\alpha = f_m \otimes e_n$ where $m=\alpha-n,\ n=0,1,\ldots$. Note that

$$\|J_n^\alpha\|^2 \equiv \langle J_n^\alpha, J_n^\alpha\rangle = (-1)^{\frac{1}{2}(\alpha-n)(\alpha-n+1)}. \tag{107}$$

Applying C to the orthogonal set $\{J_n^\alpha\}$ we find

$$\begin{aligned} CJ_n^\alpha &= -q^{\frac{1}{2}(n-\alpha+1)}\omega\left[\frac{(1-p^{n+1})u}{1+q}\right]^{\frac{1}{2}} J_{n+1}^\alpha + (-1)^{n+\alpha}q^{\frac{1}{2}(n-\alpha)}\omega\left[\frac{(1-p^n)u}{1+q}\right]^{\frac{1}{2}} J_{n-1}^\alpha \\ &\quad + p^n\left[-q^{2\lambda-\alpha+1}u + (-1)^\alpha\omega^2 q^{-2\lambda}\right] J_n^\alpha. \end{aligned} \tag{108}$$

The operator C is self-adjoint. If we introduce the spectral transform of this operator so that C corresponds to multiplication by the transform variable x, then (108) takes the form of a three-term recurrence relation for orthogonal polynomials $J_n^\alpha(x)$ of order m in x. Indeed, comparing (108) with the three-term recurrence relation for the polynomials

$$P_n(x,z) = {}_2\phi_1\left(\begin{matrix} Q^{-n}, & 1/x \\ 0 & \end{matrix}; Q, -xQ/z\right) \tag{109}$$

$$xP_n = zQ^nP_{n+1} + (1-z)Q^nP_n + (1-Q^n)P_{n-1} \tag{110}$$

we see that we get a match provided $Q=p,\ z_\alpha = q^{4\lambda+1}u/(1+q)\omega^2 p^\alpha$ and $C \sim x\omega^2 q^{-2\lambda}(-1)^\alpha$. Making the identification

$$J_n^\alpha(x,s) = \frac{q^{n(n-1)/4}}{\sqrt{(p;p)_n}}\left[\frac{q^{4\lambda+1}u}{(1+q)\omega^2 q^\alpha}\right]^{n/2} P_n(x,z_\alpha)s^\alpha(-1)^{n(n+1)/2} \tag{111}$$

we have

$$\langle J_n^\alpha, J_{n'}^{\alpha'}\rangle' = \delta_{\alpha\alpha'}\delta_{nn'}(-1)^{(\alpha-n)(\alpha-n+1)/2} \tag{112}$$

where

$$\begin{aligned} \langle f(x)s^\alpha, g(x)s^{\alpha'}\rangle' &\equiv \delta_{\alpha\alpha'}\frac{(-1)^{\alpha(\alpha+1)/2}(1+z_\alpha)}{z_\alpha(p,-1/z_\alpha,-z_\alpha;p)_\infty} \\ &\quad\times\left(\sum_{k=0,x=p^k}^\infty x(px;p)_\infty\left(-\frac{xp}{z_\alpha};p\right)_\infty f(x)g(x)\right. \\ &\quad\left. - \sum_{k=0,x=-z_\alpha p^k}^\infty x(px;p)_\infty\left(-\frac{xp}{z_\alpha};p\right)_\infty f(x)g(x)\right). \end{aligned} \tag{113}$$

This is just a bilinear product, since the discrete measure can be negative. It is not difficult to verify that on this basis the superalgebra acts as follows:

$$\begin{aligned} F_+ &= \omega q^{-\lambda}s\left(x - \frac{q^{4\lambda}u}{(1+q)\omega^2}T_s^{-1}R_s\right) \qquad F_- = q^{-\lambda}\frac{\omega}{s}R_s \\ L &= -\lambda + \tfrac{1}{2}s\partial_s \end{aligned} \tag{114}$$

where $R_sG(s) = G(-s)$.

For fixed $x = z_0p^{-\alpha_0}$ this representation is equivalent to the irreducible representation $\uparrow''_{\lambda-\alpha_0/2}$ of $[0,-u]$, as given in (101). For fixed $x=p^k$ the representation is equivalent to an irreducible representation $R_k(\omega^2, q^\lambda)$, unbounded above and below. Thus we have derived the decomposition

$$\omega'[0,0] \otimes {\uparrow''}_\lambda[0,-u] \cong \oplus \sum_{\alpha_0=-\infty}^{\infty} {\uparrow}''_{\lambda-\alpha_0/2}\,[0,-u] \oplus \sum_{k=0}^{\infty} R_k(\omega^2, q^\lambda)[0,-u]. \tag{115}$$

Here the reduced basis corresponding to the representation ${\uparrow}''_{\lambda-\alpha_0/2}$, equation (101), is given by

$$e_n^{(\alpha_0)}(x,s) = (-1)^{n(n+1)/2+\alpha_0 n}\,\frac{\delta(x+z_{\alpha_0})}{q^{n(n+1)/4}}\sqrt{(p;\,p)_n(-z_{\alpha_0}^{-1};\,p)_\infty |z_{\alpha_0}|^n}\; s^\alpha. \tag{116}$$

We have

$$\langle e_n^{(\alpha_0)}, e_{n'}^{(\alpha_0')}\rangle'' = \delta_{nn'}\delta_{\alpha_0\alpha_0'}(-1)^{\alpha_0(\alpha_0+1)/2}\ \mathrm{sign}\,(-z_{\alpha_0}^{-1};\,p)_\infty.$$

The reduced basis corresponding to representation $R_k(\omega^2, q^\lambda)$ is given by

$$g_\alpha^{(k)}(x,s) = \frac{\delta(x-p^k)q^{k(k-1)/4}}{|z_0|^{k/2}q^{-\alpha k/2}}\sqrt{(p;\,p)_k|(-z_\alpha p^{-k};\,p)_\infty|} \tag{117}$$

where

$$\langle g_\alpha^{(k)}, g_{\alpha'}^{(k')}\rangle'' = \delta_{\alpha\alpha'}\delta_{kk'}\frac{(-1)^{k+(\alpha+k)(\alpha+k+1)/2}}{\mathrm{sign}\,(-z_\alpha p^{-k};\,p)_\infty}.$$

The superalgebra action is

$$\begin{aligned}
F_+ g_\alpha^{(k)} &= (-1)^k q^{k/2}\sqrt{\left|\omega^2 q^{-2\lambda} - \frac{q^{2\lambda}p^{-k-\alpha}}{1+q}\right|}\ \mathrm{sign}\left(\omega^2 - \frac{q^{4\lambda}p^{-k-\alpha}}{1+q}\right) g_{\alpha+1}^{(k)} \\
F_- g_\alpha^{(k)} &= (-1)^\alpha q^{k/2}\sqrt{\left|\omega^2 q^{-2\lambda} + \frac{q^{2\lambda+1}p^{-k-\alpha}}{1+q}\right|}\ g_{\alpha-1}^{(k)} \\
L g_\alpha^{(k)} &= \left(-\lambda + \tfrac{1}{2}\alpha\right) g_\alpha^{(k)}
\end{aligned} \tag{118}$$

where $k, \pm\alpha = 0, 1, \ldots$.

Again, the functions $J_n^\alpha(x,s)$ are the Clebsch–Gordan coefficients for this decomposition. As before, the decomposition (115) can be used to obtain an identity relating the matrix elements of the 'group' operators with respect to the tensor product and the reduced bases.

Acknowledgment

WM acknowledges partial financial support by the National Science Foundation under grant DMS 94-00533.

References

[1] Agarwal A K, Kalnins E G and Miller W 1987 Canonical equations and symmetry techniques for q-series *SIAM J. Math. Anal.* **18** 1519–38

[2] Bars I 1985 Supergroups and their representations *Supersymmetry in Physics* ed V A Kostelecký and D K Campbell (Amsterdam: North Holland) pp 107–84

[3] Chaichian M and Kulish P 1990 Quantum Lie superalgebras and q-oscillators *Phys. Lett.* **234B** 72–80

[4] Chung W S, Kalnins E G and Miller W Tensor products of q-superalgebras and q-series identities. I, submitted

[5] Floreanini R and Vinet L 1991 Quantum algebras and q-special functions *Lett. Math. Phys.* **22** 45–54

[6] Gasper G and Rahman M 1990 *Basic Hypergeometric Series* (Cambridge: Cambridge University Press)

[7] Granovskii Ya I and Zhedanov A S 1994 Hidden symmetry of the Racah and Clebsch–Gordan problems for the quantum algebra $sl_q(2)$ *J. Group Theor. Methods Phys.*
[8] Granovskii Ya I, Zhedanov A S and Grakhovskaya O B 1992 *Phys. Lett.* **85B** 278
[9] Groza V A, Kachurik I I and Klimyk A U 1990 On matrix elements and Clebsch–Gordan coefficients of the quantum algebra $U_q(SU_2)$ *J. Math. Phys.* **31** 2769–80
[10] Jimbo M 1985 A q-difference analogue of $U(g)$ and the Yang-Baxter equation *Lett. Math. Phys.* **10** 63–9
[11] Kalnins E G and Miller W 1996 q-algebra representations of the Euclidean, pseudo-Euclidean and oscillator algebras, and their tensor products *Proc. Workshop on Symmetries of Difference Equations* (Montreal: CRM)
[12] Kalnins E G and Miller W 1996 A note on tensor products of q-algebra representations and orthogonal polynomials *J. Computat. Appl. Math.*
[13] Kalnins E G, Manocha H L and Miller W 1992 Models of q-algebra representations: tensor products of special unitary and oscillator algebras *J. Math. Phys.* **33** 2365–83
[14] Kalnins E G, Mukherjee S and Miller W 1993 Models of q-algebra representations: matrix elements of the q-oscillator algebra *J. Math. Phys.* **34** 5333–56
[15] Kalnins E G, Mukherjee S and Miller W 1994 Models of q-algebra representations: matrix elements of $U_q(su_2)$ *Lie Algebras, Cohomology and New Applications to Quantum Mechanics (Contemporary Mathematics Series)* (Providence, RI: American Mathematical Society)
[16] Kalnins E G, Mukherjee S and Miller W 1994 Models of q-algebra representations: The group of plane motions *SIAM J. Math. Anal.* **25** 513–27
[17] Koelink H T 1991 On quantum groups and q-special functions *Thesis* University of Leiden
[18] Koelink H T and Koornwinder T H 1989 The Clebsch–Gordan coefficients for the quantum group $S_\mu U(2)$ and q-Hahn polynomials *Proc. Koninklijke Nederl. Akad. Wetenschappen* A **92** 443–56
[19] Koornwinder T H 1989 Representations of the twisted $SU(2)$ quantum group and some q-hypergeometric orthogonal polynomials *Nederl. Akad. Wetensch. Proc.* A **92** 97–117
[20] Kostelecky V A, Nieto M M and Truax D R 1986 Baker–Campbell–Hausdorff relations for supergroups *J. Math. Phys.* **27** 1419–29
[21] LeBlanc R and Rowe D J 1990 Superfield and matrix realizations of highest weight representations for $osp(m/2n)$ *J. Math. Phys.* **31** 14–36
[22] Miller W 1970 Lie theory and q-difference equations *SIAM J. Math. Anal.* **1** 171–88
[23] Saleur H 1990 Quantum $osp(1, 2)$ and solutions of the graded Yang–Baxter equation *Nucl. Phys.* B **336** 363–76
[24] Swarttouw R F 1992 The Hahn–Exton q-Bessel function *Thesis* Delft University of Technology pp 97–117
[25] Truax D R, Kostelecky V A and Nieto M M 1986 On the theory of one-parameter subgroups of supergroups *J. Math. Phys.* **27** 354–64
[26] Vaksman L L and Korogodskiĭ L I 1989 An algebra of bounded functions on the quantum group of the motions of the plane, and q-analogues of Bessel functions *Sov. Math. Dokl.* **39** 173–7
[27] Vilenkin N Ja and Klimyk A U 1992 *Representations of Lie Groups and Special Functions* vol 3 (Russian translation) (Dordrecht: Kluwer) ch 14
[28] Zhedanov A S 1993 Q rotations and other Q transformations as unitary nonlinear automorphisms of quantum algebras *J. Math. Phys.* **34** 2631–47